Fluctuating Fronts as Correlated Extreme Value Problems: An Example of Gaussian Statistics

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In this paper, we view fluctuating fronts made of particles on a one-dimensional lattice as an extreme value problem. The idea is to denote the configuration for a single front realization at time t by the set of co-ordinates $\{k_i(t)\} \equiv [k_1(t), k_2(t), \ldots, k_{N(t)}(t)]$ of the constituent particles, where $N(t)$ is the total number of particles in that realization at time t. When $\{k_i(t)\}$ are arranged in the ascending order of magnitudes, the instantaneous front position can be denoted by the location of the rightmost particle, i.e., by the extremal value $k_f(t) = \max[k_1(t), k_2(t), \ldots, k_{N(t)}(t)]$. Due to interparticle interactions, $\{k_i(t)\}$ at two different times for a single front realization are naturally not independent of each other, and thus the probability distribution $P_{k_f}(t)$ [based on an ensemble of such front realizations] describes extreme value statistics for a set of correlated random variables. In view of the fact that exact results for correlated extreme value statistics are rather rare, here we show that for a fermionic front model in a reaction-diffusion system, $P_{k_f}(t)$ is Gaussian. In a bosonic front model however, we observe small deviations from the Gaussian.

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I. INTRODUCTION

Extreme value statistics of random variables plays a diverse role in physics, chemistry and biology [1, 2, 3]. The topic concerns the probability distributions of the extrema (i.e., the maximum k_{max} or the minimum k_{min}) of a set of N random variables $\{k_1, k_2, \ldots, k_N\}$ in the limit $N \to \infty$. When the random variables k_i are uncorrelated, the probability distribution of k_{min} belongs to one of the three universality classes [4], but the identification of similar universality classes for the extreme value statistics of correlated random variables is still largely an open problem. A few results relating to extreme value statistics for correlated random variables in physics, computer science and mathematics have been obtained in the recent past [5, 6]; nevertheless, any exact result that can be obtained for correlated random variables is an important addition to the present state of knowledge.

From this perspective, in this paper, we present two main results relating to fluctuating fronts made of discrete particles on a one-dimensional lattice. Before we proceed further with our formulation of the problem, we must note that an intriguing connection between the extreme value statistics of correlated random variables and travelling fronts have already emerged from the recent works [6, 7]. To be more precise, these works have demonstrated, for the models they studied, that the cumulative probability distributions of extreme values for correlated random variables admit propagating front solutions, wherein the variance of the extremal variable is the front width itself. Our formulation here, however, is completely the other way round: namely that our systems consist of many *interacting* particles, where the dynamics of the systems *already* admits front solutions propagating into unstable states. Although in a deterministic mean-field description, these fronts propagate with a fixed speed and a fixed shape at long times, due to the presence of stochasticity involving many particles, the front in a given realization of the system does not move with a uniform speed even at long times — instead, the front speed averaged over an ensemble of front realizations approaches a constant in time at long times. Moreover, as a result of the inherent stochasticity in these systems, the individual front realizations that are initially aligned with each other do not remain so at a later time; instead their displacement w.r.t. each other keeps increasing with time (see Fig. 4 of Ref. [8] for an illustration). As explained below, it is the dynamics of the individual front realizations in the ensemble that we pose as a correlated extreme value problem in this paper.

The correspondence between the extremal value statistics and the fluctuating fronts in these systems is easily made by first noticing that in any realization of these systems, the front position can be denoted by the instantaneous position $k_f(t)$ of the foremost (or the rightmost) particle [9, 10]. The interest then lies in the probability distribution $P_{k_f}(t)$, which describes the statistics of the front position in time for an ensemble of front realizations. Secondly, in a snapshot of one single realization, the configuration of the system is described by the locations of the particles (as random variables) $\{k_i(t)\} \equiv [k_1(t), k_2(t), \ldots, k_{N(t)}(t)]$, where $N(t)$ is the total number of particles in that realization at time t. Then the instantaneous front position $k_f(t)$ in this formulation is then simply the extremal value $\max[k_1(t), k_2(t), \ldots, k_{N(t)}(t)]$. Due to the interparticle interaction within the system defined by the microscopic rules of the dynamics, $\{k_i(t)\}$ are naturally not independent of each other, and thus $P_{k_f}(t)$ simply describes the statistics of the extreme for a set of correlated random variables.

In this paper, we consider two different systems that admit front solutions propagating into unstable states:

(a) a fermionic reaction-diffusion system $A \rightleftharpoons A+A$ [10, 11, 12] in Sec. II, where we show that $P_{k_f}(t)$ is Gaussian, and (b) the so-called (bosonic) clock model [13] in Sec. III, where $P_{k_f}(t)$ has small deviations from the Gaussian. It is important to note here that the front solutions in these models have been analyzed before, in the sense that both the front speed $v = \lim_{t \to \infty} d\langle k_f(t) \rangle / dt$ and the front diffusion coefficient $D_f = \lim_{t \to \infty} d\langle [k_f(t) - vt]^2 \rangle / dt$, respectively based on the first and the second moments of $P_{k_f}(t)$, have previously been analyzed and numerically measured [8, 10, 11, 12, 13, 14]. The higher (than second) moment of $P_{k_f}(t)$, or $P_{k_f}(t)$ itself, however, have not been extensively studied before.

The paper is finally ended with a discussion in Sec. IV.

II. A FERMIONIC REACTION-DIFFUSION MODEL AND GAUSSIAN BEHAVIOUR OF $P_{k_f}(t)$

In this model, we consider a one-dimensional lattice on which at most one A particle is allowed per lattice site at any instant — hence the model is named fermionic. The particles can undergo the following three basic moves, shown in Fig. 1: (i) A particle can diffuse to any one of its neighbour lattice sites with a diffusion rate D, provided this neighbouring site is empty. (ii) Any particle can give birth to another one on any one of its empty neighbour lattice site with a birth rate ε. (iii) Any one of two A particles belonging to two neighbouring filled lattice sites can get annihilated with a death rate W.
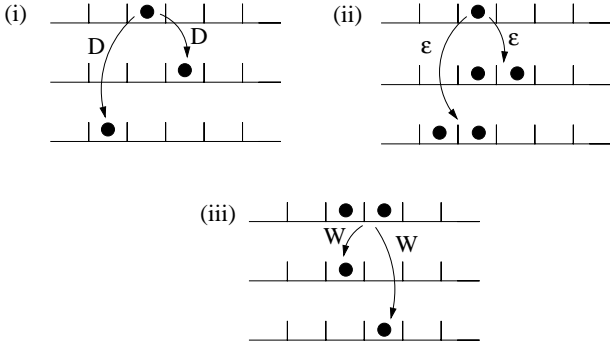

FIG. 1: The microscopic processes that take place inside the system: *(i)* A diffusive hop with rate D to a neighboring empty site; *(ii)* Creation of a new particle on a site neighboring an occupied site with rate ε; *(iii)* Annihilation of a particle on a site adjacent to an occupied site at a rate W.

The lattice indexed by k that we consider in this problem is semi-infinite. The left boundary is impenetrable — no particle can diffuse across the left boundary located on the left of the lattice site $k = 0$, while the system is of infinite extent on the right side. Following the usual convention, we start with a step initial condition, i.e., at time $t = 0$, there exists a finite k_{right}, such that all lattice sites $0 \leq k \leq k_{\text{right}}$ are occupied and $k > k_{\text{right}}$

are empty. This system then admits a fluctuating (and propagating) front solution for $t > 0$.

Earlier work on models of this type has appeared in Refs. [10, 11, 12, 15]. In the general case there are essentially only two nontrivial parameters in our model, e.g., the ratios D/ε and D/W, since an overall multiplicative factor simply sets the time scale. When these ratios tend to infinity, the front speed approaches its mean field value [11].

For an ensemble of front realizations, let us denote the probability distribution for the foremost occupied lattice site to be at lattice site k_f at time t by $P_{k_f}(t)$. The evolution of $P_{k_f}(t)$ is then described by

$$
\frac{dP_{k_f}}{dt} = (D + \varepsilon) P_{k_f-1} + \left[D\, P_{k_f+1}^{\text{empty}} + W\, P_{k_f+1}^{\text{occ}} \right]
$$
$$
- (D + \varepsilon) P_{k_f} - \left[D\, P_{k_f}^{\text{empty}} + W\, P_{k_f}^{\text{occ}} \right]. \quad (1)
$$

Here $P_{k_f}^{\text{occ}}(t)$ and $P_{k_f}^{\text{empty}}(t)$ respectively denote the joint probabilities that the foremost particle is at site k_f and that the site $k_f - 1$ is occupied and empty. Clearly, $P_{k_f}(t) = P_{k_f}^{\text{occ}}(t) + P_{k_f}^{\text{empty}}(t)$, and $\sum_{k_f} P_{k_f}(t) = 1$. The first term on the r.h.s. of Eq. (1) describes the increase in $P_{k_f}(t)$ due to the advance of a foremost occupied lattice site from position k_f-1, while the second term describes the increase in $P_{k_f}(t)$ due to the retreat of a foremost occupied lattice site from position k_f+1. The third and the fourth terms respectively describe the decrease in $P_{k_f}(t)$ due to the advance and retreat of a foremost occupied lattice site from position k_f. It is clear from this formulation that the dynamics of $P_{k_f}(t)$ is effectively obtained only from the coupled interaction between the foremost particle and the site just behind it.

In addition to Eq. (1), we have

$$
P_{k_f}^{\text{occ}}(t) = \rho_{k_f-1}(t) P_{k_f}(t), \quad (2)
$$

where $\rho_{k_f-1}(t)$ is the conditional probability of having the (k_f-1)th lattice site occupied. At large t, $\rho_{k_f-1}(t)$ should be independent of k_f and t, and one can replace $\rho_{k_f-1}(t)$ by $\bar{\rho}$ in Eq. (2), where the numerical value of $\bar{\rho}$ depends only on those of D, ε and W. Similarly, the set of (time and k_f-independent) conditional occupation densities $\rho_{k_f-m}(t)$ for $m \geq 1$ can be thought of as determining the front profile in a frame moving with the foremost particle of each front realization (see Fig. 5 of Ref. [10] for an illustration).

With the condition $P_{k_f}(t) = P_{k_f}^{\text{occ}}(t) + P_{k_f}^{\text{empty}}(t)$, and the notation $q = \bar{\rho}(W - D)$, at large t, Eq. (1) can be rewritten as

$$
\frac{dP_{k_f}}{dt} = \frac{1}{2}(2D + \varepsilon + q)\left[P_{k_f+1} + P_{k_f-1} - 2P_{k_f} \right]
$$
$$
- \frac{1}{2}(\varepsilon - q)\left[P_{k_f+1} - P_{k_f-1} \right], \quad (3)
$$

which is clearly a diffusion equation for $P_{k_f}(t)$ with a drift. After having aligned the locations of the foremost particles for all realizations in the ensemble, say

at $k_f = k_{\rm in}$ at time $t_{\rm in} \gg 1$ [i.e., $P_{k_f}(t_{\rm in}) = \delta_{k_f,k_{\rm in}}$], we are interested in the solution of $P_{k_f}(t)$. In fact, Eq. (3) can be solved by taking a discrete Fourier transform in k_f, but due to the redundancy of the wavevector modulo any multiple of 2π, the magnitude of the wavevector has to be kept confined only within the first Brillouin zone $(-\pi, \pi]$. Then for $\Delta t = t - t_{\rm in} \gg 1$, it is easily seen that the dominant contribution to $P_{k_f}(t)$ comes from the wavevector in the neighbourhood of zero, yielding [16]

$$P_{k_f}(t) = \frac{\exp\left[-\dfrac{(k_f - k_{\rm in} - v\Delta t)^2}{4D_f \Delta t}\right]}{\sqrt{4\pi D_f \Delta t}}. \qquad (4)$$

Here, $v = \varepsilon - q$ is the front speed and $D_f = 2D + \varepsilon + q$ is the front diffusion coefficient, as already derived as the first and the second moment of $P_{k_f}(t)$ in Ref. [10].

III. THE CLOCK MODEL AND THE NON-GAUSSIAN BEHAVIOUR OF $P_{k_f}(t)$

The clock model was originally invented in the context of the largest Lyapunov exponent for a gas of hard spheres [13]. In this model, one considers a system of N clocks with integer readings $\{k_i\}$. The dynamics of the clocks involve binary "collisions" between any two randomly chosen clocks in continuous time. In a collision between two clocks with pre-collisional readings k_i and k_j, the post-collisional readings of both clocks are updated to $\max[k_i, k_j] + 1$.

In the clock reading space, which can be imagined as a one-dimensional lattice, the number of clocks N_k with readings k or higher for any realization of the clock model admits a fluctuating (and propagating) front solution [13]. Clock model allows more than one clock with the same reading and hence the model is bosonic. Conventionally, all clock readings in any realization are initially (i.e., at $t = 0$), taken to be equal to zero — for the propagating front, this corresponds to the step initial condition [13]. Once again we denote the largest clock reading in any realization at time t by $k_f(t)$.

In the deterministic mean-field limit, the propagating front in the clock model is a pulled front [17], and if the time is rescaled in order to have the mean collision frequency of a single clock equal to unity, the front propagates with a speed $v^* = 4.31107\ldots$ [13]. However, due to stochasticity effects associated with discreteness effects of the clocks and their readings, in the limit of asymptotically large N, the front speed v and front diffusion coefficient D_f, which could be measured following the procedure described in the last paragraph of Sec. I, have the property that $(v^* - v) \propto 1/\ln^2 N$ and $D_f \propto 1/\ln^3 N$ [14]. Thus, the clock model is an example of a fluctuating "pulled" front [8, 9, 14].

To write a master equation for $P_{k_f}(t)$ defined over an ensemble in the clock model, it may be argued that the reading of any clock in any realization can only increase

with time; and thus, $P_{k_f}(t)$ can increase when in a realization, one of the clocks with largest reading $k_f - 1$ is involved in a collision with another one. Similarly, $P_{k_f}(t)$ can decrease when in a realization, one of the clocks with largest reading k_f is involved in a collision with another one. If we now denote the conditional probability of the number of clocks with largest reading k_f to be $n_{k_f}(t)$ at time t by $\mathcal{P}(n_{k_f}, t)$, the master equation for $P_{k_f}(t)$ reads

$$\frac{dP_{k_f}}{dt} = \left[\sum_{n_{k_f-1}} C(n_{k_f-1})\,\mathcal{P}(n_{k_f-1}, t)\right] P_{k_f-1}$$
$$- \left[\sum_{n_{k_f}} C(n_{k_f})\,\mathcal{P}(n_{k_f}, t)\right] P_{k_f}. \quad (5)$$

Here, $C(n_{k_f})$ is the rate of collisions that involve a clock with reading k_f for a realization with k_f as the largest of the clock readings. From Eq. (5), one might now further argue that at large t, the quantities within the large square brackets in Eq. (5) are independent of t and k_f, and thus at large t, Eq. (5) should reduce to a form $\dfrac{dP_{k_f}}{dt} = \bar{c}\,[P_{k_f-1} - P_{k_f}]$, where $\bar{c} = \left[\sum_{n_{k_f}} C(n_{k_f})\,\mathcal{P}(n_{k_f}, t)\right]$ at large t. However, for any finite value of N, the simple-minded replacement of $\left[\sum_{n_{k_f}} C(n_{k_f})\mathcal{P}(n_{k_f}, t)\right]$ by a t and k_f-independent quantity $\bar{c}$ in Eq. (5) at large t is incorrect for the clock model — caused by the fact that $\mathcal{P}(n_{k_f}, t)$ does *not* become time-independent at large t [20] — as we argue below.

The observation we make, in order to argue that $\mathcal{P}(n_{k_f}, t)$ does not become independent of t at large t, is that the largest clock reading in any realization does not increase smoothly in time with a rate v even at large t. Instead, after attaining a new integer value, the largest of the clock readings for any given realization does not change for some time-interval (hereafter denoted by δt) of typical magnitude $1/v$ before attaining the next integer value [18]. Generally speaking, during any of these time-intervals, the number of clocks with the largest clock reading in any realization increases with time; and the number of clocks with the largest reading at any instant in a given realization depends on how long the largest clock reading remains unchanged at its value. The conditional probability $\mathcal{P}(n_{k_f}, t)$ can thus be written as

$$\mathcal{P}(n_{k_f}, t) = \int_0^\infty d(\delta t)\,\wp_1(\delta t, t)\,\wp_2(n_{k_f}, \delta t, t), \qquad (6)$$

With Eq. (6) in the back of our minds, we now return to the statement to the second sentence of Ref. [18]: namely that front propagation in any realization of the clock model is coded in the *sequential* values of the time-intervals $\{\delta t_i\}$ between the consecutive changes of the

largest clock reading. In this description, the point to note is that the δt_i values are very strongly correlated with each other; e.g., a large δt is almost always followed by several small values of δt and vice versa (the large or smallness of δt are decided in comparison to $1/v$) [8, 9]. Due to such strong dependence of of the δt values on the evolution histories of individual realizations, it is easily conceivable that the shape of the probability distribution $\wp_1(\delta t, t)$ lacks t-independence at large t. where
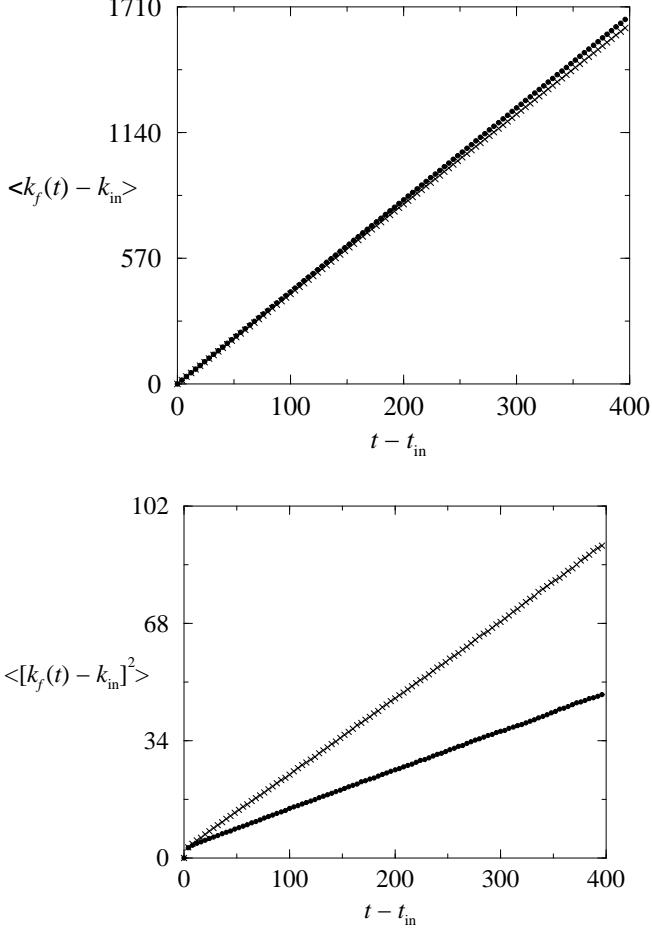

FIG. 2: Top figure: Simulation results for $\langle k_f(t) - k_{in} \rangle$ as a function of $t - t_{in}$. Bottom figure: Simulation results for $\langle [k_f(t) - k_{in}]^2 \rangle$ as a function of $t - t_{in}$. Apart from an initial transient for $t - t_{in} \lesssim 10$, $\langle [k_f(t) - k_{in}]^2 \rangle$ increases linearly with t, indicating that the front wandering is diffusive. Crosses correspond to $N = 10^4$ (front speed $v = 4.08$ and front diffusion coefficient $D_f = 0.112$) and filled circles correspond to $N = 10^5$ (front speed $v = 4.17$ and front diffusion coefficient $D_f = 0.056$) in both figures.

$\wp_1(\delta t, t)$ is the probability that the largest clock reading became k_f at time $(t - \delta t)$ and remains so until time t, and $\wp_2(n_{k_f}, \delta t)$ is the probability of having n_{k_f} clocks at time t when the largest clock reading became k_f at time $(t - \delta t)$ and remains so until time t. Using Eq. (6), the t-dependence of $\mathcal{P}(n_{k_f}, t)$ can then be argued *in terms of the t-dependences of $\wp_1(\delta t, t)$ and $\wp_2(n_{k_f}, \delta t, t)$.*

The t-dependence of $\wp_2(n_{k_f}, \delta t, t)$ can be argued in a similar way. In realizations for which the largest clock value became k_f at time $(t - \delta t)$ and remains so until time t, how many clocks share the largest clock reading at time t depends on the time-dependence of the number of clocks n_{k_f-1} with clock readings $(k_f - 1)$ between times $(t - \delta t)$ and t — after all, any clock that attains a reading k_f must come out of a collision that involves a clock with reading $(k_f - 1)$. Between times $(t - \delta t)$ and t, $n_{k_f} - 1$ changes also with time, and thus the probability distribution $\wp_2(n_{k_f}, \delta t, t)$ inherently connects to the *fluctuations in the shapes of individual front realizations* [8, 14]. These fluctuations have a typical correlation time $\propto \ln^2 N$ [8, 14, 19]. For $N \to \infty$, this correlation time also becomes large, and one therefore expects the shape of $\wp_2(n_{k_f}, \delta t, t)$ to also depend on t via the strong dependence of n_{k_f-1} on the evolution histories of individual realizations at earlier times.
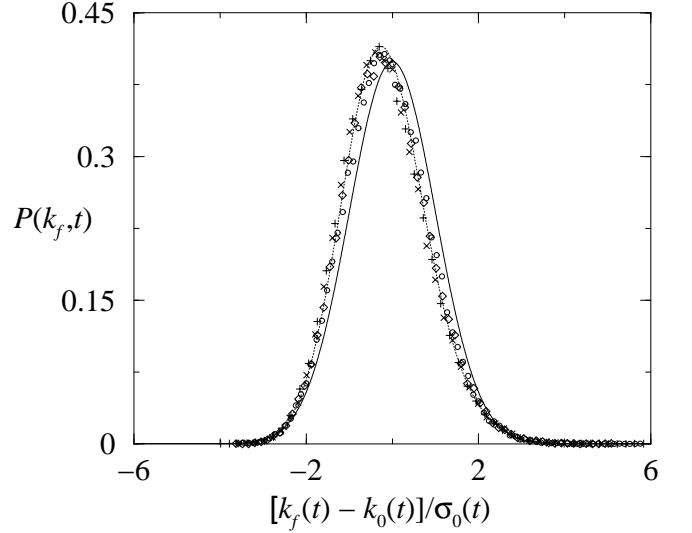

FIG. 3: $P_{k_f}(t)$ for the clock model; pluses: data for $N = 10^4$ at $t = t_{in} + 99$, circles: data for $N = 10^4$ at $t = t_{in} + 297$, crosses: data for $N = 10^5$ at $t = t_{in} + 198$ and diamonds: data for $N = 10^5$ at $t = t_{in} + 396$. Solid line: normalized Gaussian distribution with mean zero and variance unity. Dotted line: numerically obtained curve for the collapsed data.

With no further simplification of Eq. (5) possible, let alone an exact solution for $P_{k_f}(t)$ like in Eq. (4), we can only study $P_{k_f}(t)$ for the clock model only via simulation. Our (molecular dynamics) simulation methods are as follows: we choose an ensemble of $\mathcal{N} = 50000$ realizations of $N = 10^4$ clocks and set all clock values zero at $t = 0$. We then let each realization evolve until time $t_{in} = 800$ units. At t_{in}, we align the different realizations in such a way that the largest of all the clock values coincide at $k = k_{in}$. We then follow the locations of the largest clock values for each realization until $t - t_{in} = 400$. We also repeat the calculations for the same values of $\mathcal{N}$, t_{in} and k_{in} but for $N = 10^5$. The ensemble average $\langle k_f(t) - k_{in} \rangle$

and $\langle[k_f(t)-k_{\text{in}}]^2\rangle$ for $\mathcal{N}=50000$ as a function of $(t-t_{\text{in}})$ both for $N=10^4$ and 10^5 have been shown in Fig. 2.

To obtain $P_{k_f}(t)$ numerically from the above data, we now proceed in the following way. First we select two different time instants for each value of N to take snapshots of the entire ensemble of k_f-values: for $N=10^4$, we choose $t=t_{\text{in}}+99$ and $t_{\text{in}}+297$, and for $N=10^5$, we choose $t=t_{\text{in}}+198$ and $t_{\text{in}}+396$. Having used the best fit method from the data of Fig. 2, we then identify the location of the mean front position $k_0(t)$ and the standard deviation $\sigma_0(t)$ [$\sigma_0(t)$ effectively behaves as $\sim\sqrt{2D_f(t-t_{\text{in}})}$ as seen in the bottom plot of Fig. 2 for large $(t-t_{\text{in}})$], for two different values of N at these different time instants. Finally, with the histograms $\mathcal{N}(k_f,t)/[\mathcal{N}\sigma_0(t)]$ plotted as a function of $[k_f-k_0(t)]/\sigma_0(t)$, where $\mathcal{N}(k_f,t)$ is the number of realizations with largest of the clock values k_f at time t, we expect a good data collapse, and the corresponding curve then gives us the normalized $P_{k_f}(t)$. Notice that the procedure that we followed to obtain $k_0(t)$ and $\sigma_0(t)$ [and subsequently the numerical curve for $P_{k_f}(t)$] at the above time instants does not guarantee $\langle k_f(t)\rangle-k_0(t)\equiv 0$ and $\langle[k_f(t)-k_0(t)]^2\rangle\equiv\sigma_0^2(t)$; instead, the $\langle k_f(t)\rangle-k_0(t)$ and the $\langle[k_f(t)-k_0(t)]^2\rangle/\sigma_0^2(t)$ values are in fact very close to zero and unity respectively.

This data collapse is shown by means of the numerically obtained dotted curve in Fig. 3. *Further analysis of the data (not presented here) clearly shows that the dotted curve does not belong to any of the known universality classes for the extreme value statistics of uncorrelated random variables [4]; instead, it appears to resemble the normalized Gaussian distribution rather closely.* To facilitate comparison, we therefore plot $P_{k_f}(t)$ against the normalized Gaussian distribution (with mean zero and variance unity). It is clear from Fig. 3 that the dotted curve is positively skewed; direct measurement of the third cumulant from the data also confirms this positive skewness behaviour of $P_{k_f}(t)$. The most noteworthy feature is the longer right tail of the collapsed data than the left tail, implying that the probability for large positive deviation around the mean for the clock model is larger than that of large negative deviation. This is indeed consistent with positively skewed $P_{k_f}(t)$ — as stated before, $\langle k_f(t)-k_0(t)\rangle\simeq 0$ for all snapshots.

While Fig. 3 certainly provides an example of deviation from Gaussian statistics when the fluctuating front propagation is seen as a correlated extreme value problem, it also provides an interesting perspective from the point of view of fluctuating front propagation literature. As already mentioned before, clock model is an example of fluctuating "pulled" fronts, and the expression for v and the scaling for D_f due to the discrete particle stochasticity effects in the limit of asymptotically large values of N are known for the last few years. It is also known that over a time interval Δt at large t, the second moment of $P_{k_f}(t)$, i.e., $\langle[k_f(\Delta t)-k_{\text{in}}-v\Delta t]^2\rangle\sim 2D_f\Delta t$ for all values of N. Figure 3 however shows that the information regarding the second moment is clearly not

enough to characterize $P_{k_f}(t)$. Nevertheless, the data collapse shows that at large t, $P_{k_f}(t)\equiv P[(k_f-k_{\text{in}}-v\Delta t)/\sqrt{2D_f\Delta t}]/\sqrt{2D_f\Delta t}$ (i.e., the dotted line in Fig. 3) is a characteristic curve for the clock model, and this characteristic curve is not Gaussian for the values of N studied here. The statement that "the front wandering is diffusive" at any value of N for the clock model, therefore, has to be interpreted only in the sense that the second moment of $P_{k_f}(t)$ increases linearly with time at large t for any value of N.

Whether the deviation of the dotted line from the Gaussian is due to the fact that we have not used extremely large values of N is however not clear. It is well known that to observe the $1/\ln^2 N$ scaling of (v^*-v) and the $1/\ln^3 N$ scaling of D_f for fluctuating "pulled" fronts one needs to take N extremely high [13, 14]. Direct molecular dynamics simulations of the clock model for $N\gtrsim 10^6$ are prohibitively slow. The existing simulation methods at much higher values of N are not only quite intricate, but they also do not follow the exact dynamics of the model for all clocks. This particular point, therefore, is left here for further investigation in future.

IV. DISCUSSION

In this paper, we have analyzed front propagation in discrete particle systems on a one-dimensional lattice as extreme value problems. In these systems, the positions of the particles can be thought of as random variables, and these random variables under consideration are obviously strongly correlated with each other. We have seen that in the case of the fermionic reaction-diffusion model, the extreme value problem follows Gaussian statistics. It clearly does not belong to any of the classes pertaining to extreme value statistics of uncorrelated random variables. For the (bosonic) clock model however, we see that the extreme value statistics has a small deviation from the Gaussian, and additional analysis (now presented here) also clearly shows that the probability distribution does not belong to any of the known universality classes for the extreme valus statistics of uncorrelated random variables. However, due to the unavailability of any analytical tool, the characterization of this distribution has proved elusive. Whether the small deviation from the Gaussian is caused by the fact that we have not used extremely high values of N for our simulations thus remains an open question.

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[20] It is easily seen that the usage of $\bar{c}$ in Eq. (5) can only generate a front speed $d\langle k_f \rangle/dt = \bar{c}$ and a front diffusion coefficient $\frac{1}{2}\frac{\langle k_f^2 - \langle k_f \rangle^2 \rangle}{dt} = \bar{c}/2$. This clearly violates the known results for the clock model; namely that for large N, $(v^* - v) \propto 1/\ln^2 N$ and $D_f \sim 1/\ln^3 N$.